MiMA
MATHEMATIK MINERALIEN

Insights into mathematics

Booklet for the mathematics exhibition

Including links to download the programs

www.mima.museum

Welcome to the MiMa

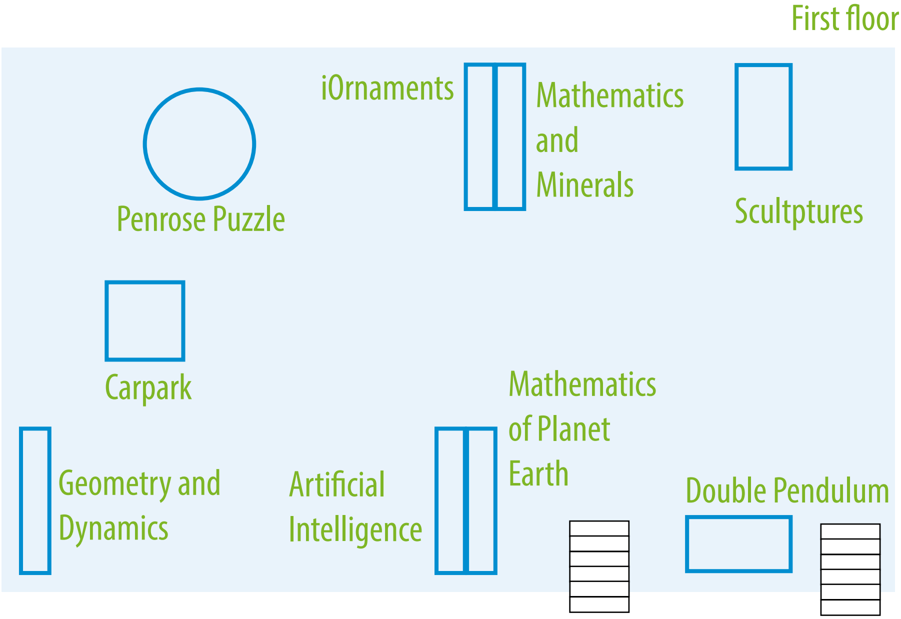
In the mathematical part of the exhibition it is deeply advisable to touch the exhibits. Only by experimenting you will discover the inherent mathematical principles.

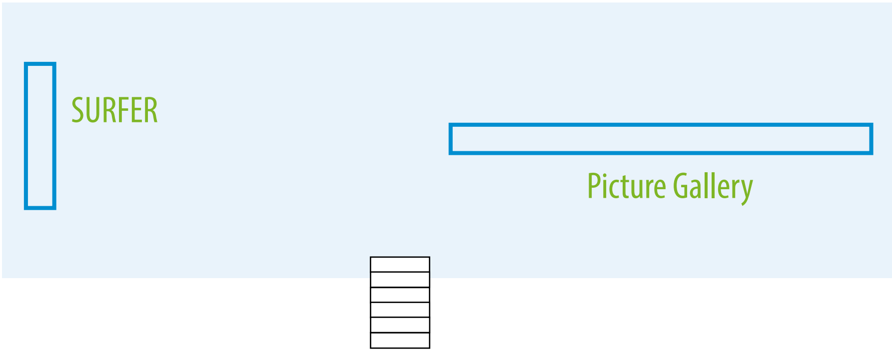
For example you can create ornaments, design curved surfaces, generate your own crystal forms or virtually fly through an infinite crystal lattice. In many exhibits you will encounter symmetry, a mathematical basic property of crystals. The museum presents objects with two-dimensional, three-dimensional, and even four-dimensional symmetries. Furthermore you will see images from different areas of mathematics, ranging from algebraic geometry and differential geometry to dynamic systems and quasicrystals.

The exhibits have been developed by mathematicians from all over the world. All programs may also be used at home. You will find links where to download them in this booklet. Further information can be found at www.mima.museum.

Enjoy your journey through the wonderful world of minerals and mathematics!

Overview


First floor
iOrnaments
Mathematics and Minerals
Penrose Puzzle
Scultptures
Carpark
Mathematics of Planet Earth
Geometry and Dynamics
Artificial Intelligence
Double Pendulum


SURFER
Picture Gallery

1 Sculptures

In the showcase you will find 3D prints of Platonic, Archimedean and Catalan solids – three-dimensional symmetrical figures that play an important role in crystallography. We also show you Platonic stars. On the lowest level you will find the projection of a four-dimensional dodecahedron into three-dimensional space.

Platonic solids

Platonic solids are perfectly regular, convex solids whose surfaces consist of equal-sized, equilateral and equiangular polygons. The same number of polygons meet in each corner. There are only five different types of these solids: tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron.

Their properties have been studied since ancient times. The Greek philosopher Plato, after whom the solids are named, described them in detail in his work Timaeus.

In crystallography, the Platonic solids play an important role because the crystals of some minerals form exact hexahedrons or octahedrons, or almost exact tetrahedrons and dodecahedrons.

Archimedean solids

The surfaces of the 13 Archimedean solids are made of different equilateral and equiangular polygons. The same number of polygons meet in each corner. For example, Archimedean solids can be constructed from the Platonic solids by cutting off the corners of the Platonic solids so that a regular polygon is formed at each corner. In nature, many Archimedean solids also occur as crystal forms.

You probably already know a very specific Archimedean solid: If a shoemaker wants to sew a soccer ball, he takes twelve regular pentagons and twenty regular hexagons. On three of the edges of a hexagon he sews another hexagon, on the other three edges he takes a pentagon instead. Mathematically, this is called a truncated icosahedron. If you cut off the corners of an icosahedron, you get exactly this shape. Try it out for yourself at the "Mathematics and Minerals" station with the "Polyhedra" program!

Catalan solids

If you take the midpoints from the faces of the Archimedean solids and connect them together you receive a Catalan solid or *Archimedean dual*. All the faces of the Catalan solids are congruent to each other, but they are not regular as their side lengths are different – exactly the opposite as in the Archimedean solids.

The Catalan solids are also very important for crystallography. For example, the rhombic dodecahedron is a typical crystal form. It occurs in minerals of the garnet group.

Platonic stars

Mathematicians from the University of Vienna calculated the formulas of algebraic surfaces appropriately named "Platonic stars". Special cusps, known to mathematicians as A_2 singularities, come to lie exactly in the corners of a Platonic solid. Like the cube has eight corners, the hexahedron star has eight cusps. And like the cube, you can turn the hexahedron star in 24 different ways without distinguishing between the starting position and the result. Platonic stars thus obey the same rules of symmetry as the Platonic solids.

Acknowledgement

The Platonic solids are 3D-plastic-prints produced by the company Voxeljet Technology. The Archimedean and Catalan solids were made by Dr. Oliver Labs. The idea for the Platonic stars came from Prof. Dr. Herwig Hauser and his team at the University of Vienna. The equations were found by Alexandra Fritz, an Austrian mathematician, among others. The Institute Forwiss at the University of Passau created the 3D data for the MiMa. The sculptures were eventually printed by Voxeljet Technology.

2 Mathematics and Minerals

At this station you will find a collection of programs with which you can carry out various experiments related to crystallography.

2a Crystal flights

Fly through grids of crystals and four-dimensional spaces. Imagine you were a pilot in a nano-jet and explore the structure of the grids from inside.

Activities

Select one of several virtual 3D flights. Three of the existing flights illustrate the atomic structure of crystals: fluorite, quartz and diamond. Tap on one of the icons, put on 3D glasses and be ready to take off.

The lever at the bottom right allows you to adjust the speed of your flight. You can influence the direction by moving your hand over the projection screen. With the "x"-button you can terminate the flight and get back to the main menu.

Of course, in reality atoms do not look like colored spheres connected with rods. That is just a simple model. However, it clearly indicates that the atoms are arranged in a lattice-like structure and that their arrangement is permanently repeated in constant direction. By trying different flight directions you can examine the different symmetry planes of the crystals.

The fourth flight is a mathematical peculiarity. You can fly through a four-dimensional space consisting of 120 three-dimensional dodecahedra. The space is self-contained, which means it has no border. If you fly out at one end, you automatically come in at the opposite side. You can think of it as if you were walking on a spherical surface: At some point you will automatically come back to the same place.

Acknowledgement

This exhibit was developed by Dr. Jeff Weeks. It is based on his program Crystal Flight, which is freely available on the internet.

For further research

www.geometrygames.org

2b Polyhedron

Polyhedrons are threedimensional shapes that are solely limited by flat surfaces. They are essential in crystallography because crystal shapes often form polyhedrons.

In the program **Platonic Solids,** you will learn more about five very special polyhedra. Platonic solids are perfectly regular, convex solids whose surfaces consist of equal-sized, equilateral and equiangular polygons. The same number of polygons meet in each corner. The program allows you to nest the shapes within each other. The innermost and the outermost shape can each be removed again. Which relationships can you find between the shapes?

The program **Polyhedra** allows you to create new polyhedra based on the Platonic solids. For example, you can virtually cut off the corners or pull out the centers of the surfaces. By tapping on the magic wand, you can set the current object as the new starting point for deformation.

Platonic solids are also the topic of the program **Folding**. It shows you how the flatplan for each solid should look like.

In the program **Archimedean solids** you get to know a further group of special polyhedra. They consist of varying equilateral and equiangular polygon surfaces. The same number of polygons meet in each corner. In this program you can examine how Archimedean solids can be constructed starting from Platonic solids.

2c Crystal structures

With specific solids it is possible to fill the three-dimensional space solely through shifted replicas of the solid. Such solids are called space-filling polyhedra. In the program **Crystal packing** one can try this for five important representatives. The built structures are the basis for the atomic arrangements in crystals.

In the program **Crystal fractals** you can create large-scale structures with fractal character. Try by placing downsized replicas of the solid at the corners of the solid. If you continue iteratively, you will discover different structures depending on the initial solid. In the case of a tetrahedron the result is a a well-known fractal: the Sierpinski tetrahedron.

An essential element for the comprehension of crystal growth is the direction of the possible boundary area. Due to the atomic arrangement in the crystal lattice specific tendencies towards a direction develop, that lie symmetric to the lattice structure. With cubic crystal lattices these are primarily the facet directions of cube, octahedron or rhombic dodecahedron. If you cut through these bodies, you obtain very realistic crystal forms. These can be viewed in more detail in the program **Intersections**.

In the program **Crystal viewer**, you can create realistic looking crystal shapes by grinding facets of selected symmetrical crystal forms.

2d Atomic lattice

The so-called structure of the "stack of oranges in the supermarket" is of particular mathematical importance. This is the sphere packing with the highest density. It is not by chance that it commonly occurs as the atomic arrangement in crystal structures. In the program **Sphere packing** these packings can be viewed and certain substructures can be explored.

The atoms in the crystal lattices have got regular arrangements. In the program **Atomic lattice** these typical lattice structures can be viewed and examined.

2e Structure building

In minerals one can often detect inclusions of a substance that look like small trees. The growth of such structures can be explained through processes of diffusion and agglomeration. In the program **Particles** simple examples of the process are presented. Imagine a particle is randomly moving through the area. If it collides with a crystallization seed, it attaches itself and the next particle goes into motion. When tapping the slider "attraction" you can set the pull of the partical towards the center of the picture.

2f The fourth dimension

Mathematicians often think in more than three dimensions. In the **Hypercube** program you can see how a 4-dimensional cube is built. The "Dimension" slider demonstrates the step-by-step construction, starting with dimension 0. The 4-dimensional hypercube has 16 vertices, 32 edges, 24 side faces and 8 side spaces. The points marked with letters are moveable.

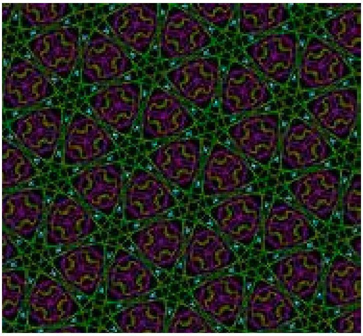
In four-dimensional space, there are a total of six highly symmetrical regular solids, analogous to the three-dimensional Platonic solids. The faces of these solids are themselves Platonic solids. In **Rotations in 4D** one can observe three-dimensional projections of these solids. The small round sliders allow rotation around six four-dimensional axes of rotation. You can adjust the projection type and decide whether the projection from four-dimensional to three-dimensional should be a parallel projection or a central projection.

To get a feeling for the transition between dimensions, you can first take a look at two-dimensional sections through three-dimensional Platonic solids in the **Plane Sections** program. The cutting plane can be adjusted using the large slider. You can select whether the section is made beginning at a corner, edge or face. Finally, the **4D Spatial Section** program enables three-dimensional sections through four-dimensional regular solids.

Acknowledgement

The programs were developed by Prof. Dr. Jürgen Richter-Gebert using Cinderella. Cinderella is a program for mathematical experiments by Prof. Dr. Jürgen Richter-Gebert and Prof. Dr. Ulrich Kortenkamp.

3 iOrnaments

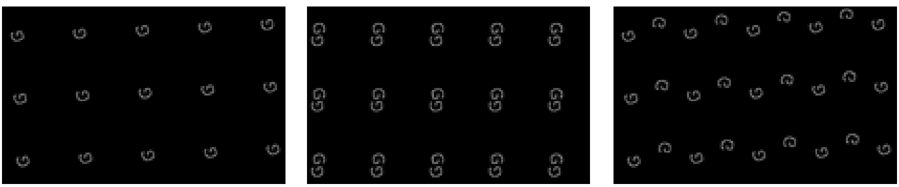


With iOrnaments you can tesselate the plane – which means cover it without gaps or overlappings – with a self-painted pattern. Wonderful ornaments with a lot of inherent mathematics are created.

Activities

Select a color from the palette at the right top. Set the size of the pencil, transparency, and color saturation with the sliders. Draw on the surface with your finger and watch how the plane gets filled with your basic pattern.

There are only 17 different possibilities to create an ornament out of your basic pattern. On the right side of the screen, below the sliders, you can choose which of these shall be applied. By pressing the first button the simplest case is applied and the basic pattern is simply shifted. By pressing the second button a half turn is performed in addition. Watch carefully what happens if you press the other buttons. More and more complex patterns result from combinations of rotations, reflections and shifts of the basic pattern.



Variations with the same basic pattern: shift, half-twist, and a combination of shift and twist.

The ornament can be continued to infinity by shifting a certain section, a "tile" again and again. You can view the tile of your ornament and also the grid with the buttons at the bottom of the menu. There you can also undo strokes or delete the complete picture to start again.

The mathematics behind

Ornaments are highly symmetric structures and therefore interesting for mathematicians. In everyday language the term symmetry is usually used as a synonym for axis or mirror symmetry. However, in mathematics rotations and shiftings are regarded as symmetric operations as well. The famous mathematician Prof. Dr. Hermann Weyl explained symmetry as follows: "A structure is symmetric if one can change it somehow and in the end receive the same result."

Look at the tile on right side of the screen once more. It contains your basic pattern in various rotated or mirrored versions. How many possibilities do you have to transform this tile into itself by rotations or reflections? Maybe you can identify one or more mirror axis or a certain point around you could rotate the tile in several steps. The set of all operations which transform the tile into itself is called a symmetry group.

One can prove mathematically, that there are exactly 17 different symmetry groups of periodic patterns or tesselations of the plane. This means, that every regular ornament on a surface belongs to exactly one of the 17 basic patterns.

The symmetry groups are very often abbreviated by letters and digits (p1, p2, pm, pg etc.), indicating the symmetric operations a group contains. For example, a digit represents a rotation center with a corresponding number of positions, an "m" stands for a mirror axis and a "g" for a glide reflection axis. The order of the letters and digits tells something about the position of the symmetries to each other.

The Arabs already knew these symmetry groups in the Middle Ages. You can find all of them in the ornaments of the Alhambra, a fortress in the Spanish city of Granada.

In English the 17 symmetry groups of periodic patterns are also called "wallpaper groups".

The link to crystallography

The 17 symmetry groups of periodic patterns are also referred to as two-dimensional crystallographic groups. In each slice plane of a crystal the atoms follow the arrangement rules of one of these 17 groups. Considering not only the slice planes, but whole, three-dimensional crystals, the situation becomes more complicated. For three-dimensional regular patterns, there are 230 crystallographic space groups.

Acknowledgement

"iOrnaments" was developed by Prof. Dr. Jürgen Richter Gebert.

For further research

www.science-to-touch.com/en/iOrnament.html

4 Penrose Puzzle

What are possible shapes of tiles, if you would like to regularly cover a floor without gaps and overlappings? A mathematical theorem states, that this can only be achieved with basic shapes which have a 1-,2-,3-,4- or 6-fold symmetry. However, with the Penrose Puzzle you can cover a plane with a pattern of figures with 5-fold symmetry. The pattern is irregular and can theoretically be continued to infinity. There is a special trick: The Penrose Puzzle uses two different basic elements.

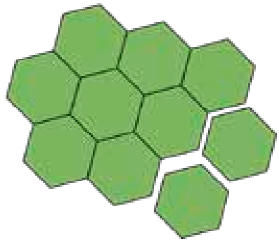
Activities

Put the pieces of the puzzle together so that the table is completely covered. Due to the shape of the table, you will not achieve a perfect match with its edge. The task is not very easy, because a once recognized pattern can not be simply repeated. The finished puzzle shows an irregular pattern, in which different symmetrical figures appear again and again.

The mathematics behind

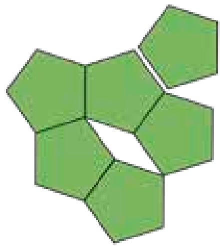
In mathematics the complete covering of a plane without gaps or overlappings is called a tesselation. The same geometrical basic element – for example a triangle or a square – is put next to each regularly and infinitely often, i.e. periodically.

A mathematical theorem states that a plane can only be tesselated with elements of 1-, 2-, 3-, 4- or 6-fold symmetry. Examine the symmetry of an element by looking for possible axes of reflection or points of rotation, where the element can be brought to coincidation with itself. For example, an equilateral triangle has a 3-fold symmetry: It is possible to rotate it three times for 120° around its center point and each time it looks the same. An equilateral pentagon has a 5-fold symmetry. According to the theorem named above it is impossible to tesselate the plane with it.



Tesselation with regular hexagons

The professional world was therefore deeply astonished, when the mathematician Prof. Dr. Roger Penrose found a tesselation of the plane with figures of 5-fold symmetry in 1974. A special trick made the impossible possible: Penrose combined two different quadrangles instead of using only one basic element. He was able to prove that the plane can be filled infinitely with these two basic elements. This produces an aperiodic pattern, in which figures of 5-fold symmetry appear again and again. With our puzzle you can reconstruct a part of such a tesselation.



Gaps appear, when trying to tesselate with pentagons

The link to crystallography

A tesselation is the two-dimensional counterpart of a crystal. In the atomic grids of crystals the same basic spatial structures are repeated again and again. If you know the basic structure, you can describe the entire structure of the crystal. You can investigate this more deeply at the station "Mathematics and Minerals".

Due to the mathematical theorem mentioned above crystallographers were convinced for a very long time, that there could be no atomic grids of crystals with 5-fold symmetry – despite the discovery of Penrose. Though, in 1982 chemist Prof. Dr. Dan Shechtman found an alloy of aluminium and magnesium with such a structure. He was awarded the Nobel Prize for chemistry in 2011.

Today, solids with this structure are known as quasicrystals. They have been explored in many ways. Among other things, they are used to produce a special type of steel, since they are particularly hard and brittle.

Acknowledgement

The Penrose-Puzzle has been built by the Mathematikum Gießen and donated to the MiMa.

5 Carpark

Carpark is an exciting logic game. Your task is to free the red cabriolet from a jam by moving the cars only forwards and backwards, but not by lifting them or moving sideways.

Activities

Above the board you will find cards with different starting situations. They are sorted by difficulty. Select one of them, build up the initial situation and then try to drive the red car out of the carpark.

The mathematics behind

Mathematicians love to solve puzzles. However, immediately they wonder how to find out if a certain problem is solvable at all and how many moves would be necessary. It is possible to construct a mathematical model of the carpark game and thus to find the possible solutions, for example the solution with the fewest moves. Such questions are considered in the fields of Optimization, Combinatorics and Game Theory, which have many applications in technology and economy.

If you would like to play this game also at home, please ask for the game "Rush Hour" in the museums shop.

Acknowledgement

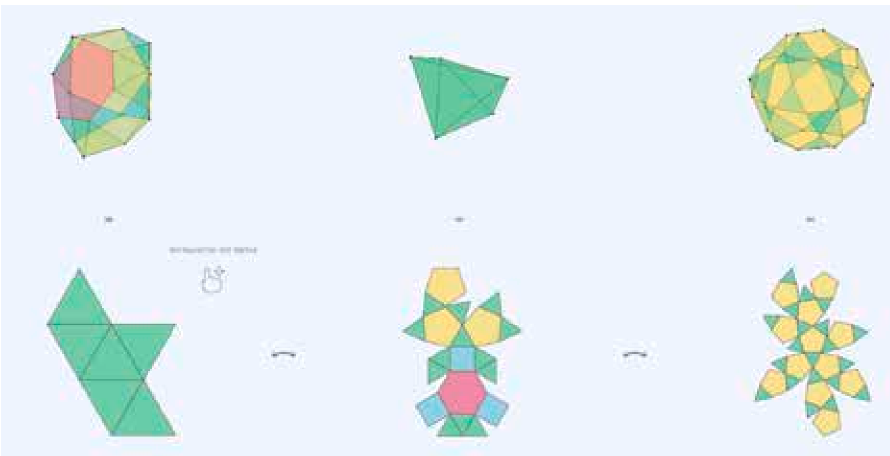
This exhibit has been built for us by the carpentry Brosemer located in Zell am Harmersbach.

6 Geometry and Dynamics

On this screen you will find a selection of programs on geometry and dynamics. You can experiment with geometrical objects, explore connections between music and geometry and work with mathematical simulations.

6a Match the Net

Do you remember how to make a cube out of a piece of paper? When a cube is "unfolded", how does it look like? The game "Match the Net" is exactly about this assignment of three-dimensional shapes to their corresponding unfolded polytopes. However, the shapes are much more complicated than a cube and thus the game quickly becomes a tricky and exciting puzzle.



Activities

Solving the tricky tasks is fun – on your own or together with others. On the start page you can choose the language, the level of difficulty and the number of polytopes. Click on "Start game". Planar folds of polytopes appear at the lower half of the screen which need to be mapped to their three-dimensional equivalents at the top half of the screen. The folds can be moved and swapped by simply pulling them over each other. Submit to check the result. Every game has several rounds. At the end the points are added together. But watch out: time also matters!

The mathematics behind

Polytopes, polygons, polyhedra – this is what geometers are dealing with. Polytopes are geometric objects with "flat" sides. Two-dimensional polytopes are called polygons. Polygons are flat structures whose overall lines are always straight and closed, even if the polygon may look crooked. Simple examples of polygons are triangles or squares. Three-dimensional polytopes are called polyhedra. These include prisms, cuboids or cubes. The sides of polyhedra are composed of polygons. For example, a pyramid may consist of a quadrilateral and four triangles.

It is an open question whether each polyhedron can be unfolded to a planar net. Of course, scientists do not use scissors and paper to solve this question. They work with computer programs like for example "polymake" to generate and unfold polytopes. These also help to find the largest possible number of nets for a polytope. There are often several ways to unfold.

Among the polyhedra there are special groups, such as the Platonic solids. They can be discovered in the game, but they can also be viewed as three-dimensional sculptures in the further exhibition.

Acknowledgement

Concept and development: Michael Joswig, Georg Loho, Benjamin Lorenz, Rico Raber and the polymake-Team (www.polymake.org/doku.php/team).

For further research

https://www.matchthenet.de/
https://imaginary.org/de/node/1168

6b FrozenLight

This game combines art, math and cryptography. A ray of light is directed through a grid of round mirrors and refracted according to the rules of ray optics. The positioning of the light source is crucial for the creation of chaos or symmetry.

Activities

With the many different buttons on the main screen you can select different reflection patterns. The top row of buttons represents the size of the grid, for instance 5 x 5 produces a grid of 5 by 5 mirrors. The percentage in the bottom row indicates the maximum radius of the mirrors for which the preset pattern works. Move the grid with your finger or drag the light source to other positions and observe the changes in the pattern.

Some buttons are different: With the ?-Button at the top left you can see the part of the light ray that is responsible for the symmetrical pattern. The percentage button on the top right allows you to change the radius of the "mirrors" for each pattern. In the lower left corner of the screen you will find an icon to adjust the colors. If you press the button in the lower right corner, an animation will show you different symmetrical patterns in succession.

The mathematics behind

The software calculates all reflections according to the laws of geometrical optics (also ray optics). The basic assumption is that the light propagates as a ray. Thus one can regard the rays as straight lines and apply the rules of geometry, such as Fermat's principle, according to which light always chooses the fastest but not necessarily the shortest path. The model may seem simple, but by experimenting with it, you will soon learn that the slightest change in the position of the light source may cause great chaos.

The software performs calculations in long-term arithmetic. These are calculations with a very high number of decimal places, which at this speed can only be solved by a computer and in turn they are limited by the performance of the computer.

Cryptographic encryption is also possible with this program. Text messages can be programmed as reflection patterns and decoded with a special symbol key.

Acknowledgement

FroZenLight has been developed by Zoltan Palmer.

For further research

frozenlight.altervista.org
www.imaginary.org/de/node/513

6c Qi

Qi is a program for interactive visualization of mathematical surface areas from the research area of differential geometry.

Activities

You can select different surface areas with the buttons on the left-hand side. The "i"-button in the above right corner gives you a short description about the chosen area. The other buttons in the menu bar on the right-hand side allow you to change the features of the individual surface areas, such as the color or the texture. The button S3 allows you to switch between the so-called Möbius transformation and the standard rotation of the area. You can turn the surface area by touching the area in its center.

The mathematics behind

Differential geometry deals with the curvature of forms. The purpose is to describe the properties of curves, curved surfaces or even shapes with more than three dimensions precisely. The fields of application of differential geometry include, for example, cartography, hydro- and aerodynamics, the calculation of flight paths, technical procedures for plastic deformation, but also the general relativity theory of Albert Einstein.

Surfaces which are optimal in the sense that a precisely defined "quality measure" can not be improved if small changes are made to the surface are particularly interesting. These include, for example, minimal surfaces. Minimal surfaces have the same properties as soap bubbles. A soapy skin, obtained by dipping a bent wire into soap soapsuds, has the smallest surface among all possible surfaces which can be clamped into the wire. Their so-called "mean curvature" is equal to zero at each point of the surface. Since the 1960s, minimal surfaces have been used as models for lightweight roofs, for example for the roof of the Olympic stadium in Munich.

Acknowledgement

Qi was developed by the GeometrieWerkstatt in the mathematical institute of the Universität Tübingen in cooperation with Nicholas Schmitt and Ulf Wagner.

For further research

www.imaginary.org/de/node/515

6d Math to Touch

Math to touch is a collection of 11 applications facilitating an easy approach to mathematics in a fun and visually engaging way. They can be accessed in the two lower rows of the menu bar. Let fate compose a new Mozart Sonata or discover and influence the swarm behavior of a school of fish!

Activities

Every program has a different, intuitive interaction design: watch out for descriptions, buttons or dots that can be moved by touch control. The arrow in the lower left corner leads you back to the main menu.

Acknowledgement

All applications were developed by Jürgen Richter-Gebert of the Technische Universität München.

For further research

www.science-to-touch.com

7 Artificial Intelligence

Artificial intelligence (AI) is considered one of the key technologies for our future society. Computer programs imitate human abilities such as logical thinking, learning, planning and creativity. But how does that actually work? How can a machine learn? Like in many others, the foundations for this technology are in mathematics. This exhibit allows to explore some basic mechanisms and principles.

7a Gradient Descent

Activities

Find the treasure chest hidden in the deepest point of the ocean floor by lowering a probe from your research ship. Use the arrow buttons to move the ship and the circle button to send the probe. What is your strategy to find the treasure as quickly as possible?

The mathematics behind

The best way to find the deepest point of the seabed is to pay attention to the gradient: How steep was the floor at the examined places and in which direction was it inclined? Although you don't see the entire seabed, the slope suggests whereto continue your search: going down the steepest slope is most promising.

This is exactly how an AI learns – namely through a mathematical algorithm called gradient descent. Similar to how we search for the deepest point of the seabed, the AI searches for the deepest point of an error function.

Training an AI means having it calculate a result from example inputs and then to give feedback on how large the error was between the expected and the calculated result. The AI now tries to reduce the error by adjusting its calculation coefficients. In doing so, it takes the gradient, i.e. the slope of the error function, into account. In which direction does the error become smaller and in which direction does it increase? If the AI is trained with sufficient sample data, it can adjust its calculation coefficients over time so that the error gets minimal. In practice, this usually happens with millions of sample data sets. Afterwards the AI can work on "real" data.

Both in the treasure hunt and in training an AI a certain difficulty arises: if you are unlucky, you will only find one deep point that is separated from the deepest point by one or more hills. You end up at a "local minimum" instead of the "absolute minimum".

If an AI finds a local minimum it gets stuck in its learning process. But there is a strategy for moving forward anyway and it’s about the same as you might use in the treasure hunt: try something different, pick a random new starting point and search again.

For further research

https://www.imaginary.org/program/gradient-descent

7b Neural Numbers

Correctly recognizing shapes and patterns is a common task for artificial intelligence. This example is about handwritten numbers. Handwriting is particularly challenging because it is inherently irregular. Characters change shape even when written by the same person and small changes can be enough to turn a 1 into a 7 or a 3 into an 8.

Activities

Write a number between 0 and 9 on the screen with your finger. An artificial neural network recognizes in real time which digit is meant. Which number does the computer recognize best? Which ones does he have problems with?

Try out what happens when you train the neural network or when you change the properties of the neural network.

The mathematics behind

An artificial neural network is an algorithm inspired by biological brains and mathematically imitates the behavior of many interconnected neurons. Each neuron receives, processes and, if necessary, forwards information. These tasks are modeled by mathematical functions.

In our example we have 784 input values (i.e. number of pixels of each processed image), 100 artificial neurons and 10 possible output values (numbers from 0-9). Each neuron takes a certain number of input values, multiplies them with so-called weights, sums everything and adds a bias. This value is then compared with a given function, the activation function. It decides which value is sent to the neuron's output. For example, the function could be defined so that positive values are passed directly, whereas negative values will lead to zero.

An artificial neural network learns from examples. It "knows" nothing at the beginning. Only by training with a lot of sample data it can adjust the weights and biases of the individual neurons at each training step and achieve increasingly precise results. In our example, the training data consists of 70,000 images of handwritten digits and the information about which digit is meant.

For further research

https://www.imaginary.org/program/neural-numbers

7c Sumory

Which strategy is best if you start knowing nothing at all? That's the idea behind the very simple game Sumory – and a question that artificial intelligence developers must deal with.

Activities

Flip the cards to reveal numbers. You have 10 turns to reach the highest possible sum. How do you proceed? Do you try as many cards as possible or do you always use the same card once you found a high positive value?

The mathematics behind

"Exploration vs. Exploitation" is a key concept for enabling learning in unfamiliar environments. If an artificial intelligence is to find a solution in an unknown environment, it must first explore the environment. At a certain point, it will use the acquired knowledge to come to a solution – until the situation changes and it randomly has to try something new again.

There are various mathematical algorithms to help an artificial intelligence maintain the balance between exploration and exploitation. Games are an excellent research field for developing and testing such algorithms. You may have heard of the computer program AlphaGo which in 2016 managed to defeat one of the best Go players in the world. Go is considered one of the most complex board games in the world. On a playing field with 19*19 fields there are more than 2*10170 possible game states. Go is therefore much more complex than chess, where it is estimated that there are around 1046 possible game states. The developers of AlphaGo used a so-called "Monte Carlo Tree Search" method, which balances "exploration vs. exploitation" very well.

For further research

https://www.imaginary.org/program/sumory

7d Reinforcement Learning

Activities

Design a maze and let the robot search for the exit. You can use obstacles to make the task more difficult and you can use rewards to encourage the robot to walk in certain direction. Watch how the robot learns. How is exploration compared to repeating paths it already knows?

The mathematics behind

Reinforcement learning refers to a range of methods in which an artificial intelligence learns to solve a task through repeated trial and error and receiving feedback from its environment. The basic idea is that the AI tries to maximize the positive feedback or rewards.

In contrast to learning with training data, the AI is not given examples of what a right or wrong action is. It must acquire this knowledge independently through interaction with its environment. Thus a form of human learning is imitated, which is known as conditioning in psychology: correct behavior is reinforced by reward, incorrect behavior is weakened by punishment.

However, a dilemma arises: How often should new actions be tried? How long should successful actions be repeated? The development of powerful algorithms to solve this dilemma is an active field of research.

For further research

https://www.imaginary.org/program/reinforcement-learning

Acknowledgement

The exhibits on artificial intelligence were developed by IMAGINARY gGmbH with the support of the Carl Zeiss Foundation and made freely available under the MIT (Expat) license.

8 Mathematics of Planet Earth

"Mathematics of Planet Earth" (MPE) is an international project in which interactive exhibits are created in competitions. Interested parties are invited to submit their designs that clearly illustrate the importance of mathematics for solving global problems. In addition to IMAGINARY, Unesco, the International Mathematical Union and the International Commission on Mathematical Instruction are organizers and partners of the competition. This station shows a selection of the best interactive applications.

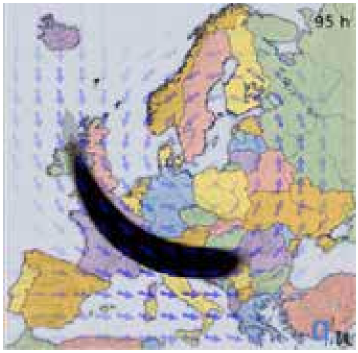
8a Dune Ash

In May 2011 the volcano Grimsvötn in Iceland erupted. Due to the resulting ash cloud the airspace in Scandinavia and Scotland was blocked and the complete airspace throughout Europe was affected. The Dune Ash program simulates how such an ash cloud spreads. You can determine the location of the volcano as well as the wind conditions and observe how the ash spreads across Europe beginning from the moment of the eruption.

Activities

Start the program by tapping on the appropriate item on the screen. Position a volcano somewhere in Europe by tapping on the map. Use the arrow button to move on to the following steps.

Next, sketch a wind field by drawing lines on the screen. The wind speed results from the "painting speed". Based on the lines you draw, a wind field is calculated and displayed across the map of Europe. The blue arrows give an impression of the wind speeds: a bright color indicates low speed, a dark color indicates high speed. The wind field remains constant for the duration of the simulation. Of course, the winds could change in reality.



The spreading of the ash is additionally affected by diffusivity, which means the random movements of the ash particles due to their thermal energy. To determine the influence of the diffusion in the simulation, set the slider in the last step.

Afterwards the program calculates the spread of the ash cloud and displays it on the screen. With "Start / Pause" the simulation can be paused, "Stop" terminates the simulation. The slider at the bottom can be used to show the situation at different times.

The mathematics behind

Simulations of complex physical processes are an important area of research in applied mathematics. They are required, for example, for weather forecasting or for the calculation of the spread of pollutants in the atmosphere. Such processes are usually influenced by many different parameters.

Mathematicians first describe the models of these processes by systems of differential equations with several variables. The equation systems become more and more complex the closer one approaches reality and the more parameters are taken into account. To solve the equation systems with mathematical algorithms on a computer is a challenging task. Very frequently, the solutions can only be determined by approximation.

The partial differential equation to be solved for the simulation of the ash cloud is:

$$\partial_t c + \nabla \cdot (\omega\, c) - \varepsilon \Delta c = \upsilon(t, x)$$

$\partial_t c$ describes the change of the concentration of the ash over time. The concentration of the ash is affected by several parameters. The term $\nabla \cdot (\omega\, c)$ stands for the influence of the wind and $\varepsilon \Delta c$ for the influence of diffusion. The function $\upsilon(t, x)$ is used to model the volcanic eruption occuring at a time t and at a location x.

Acknlowledgement

Dune Ash was developed at the Institute of Mathematics at the University of Freiburg supervised by Prof. Dr. Dietmar Kröner. The program uses the modular program package "Dune" for the solution of partial differential equations.

For further research

www.imaginary.org/program/dune-ash
www.dune-project.org

8b Future of the Glaciers

The Alpine glaciers have been shrinking for more than a century. This trend is expected to continue as long as global warming continues. But how can you make realistic predictions about the development of glaciers? The film shows how mathematicians work together with glaciologists.

Acknowledgement

The film "Future of the Glaciers" was produced at the Institute for Mathematics at the Freie Universität Berlin by Dr. Guillaume Jouvet, Dr. Chantal Landry, Dr. Antonia Mey, Prof. Dr. Marco Picasso, Prof. Dr. Jaques Rappaz, Dr. Mathias Huss, Prof. Dr. Heinz Blatter and Prof. Dr. Martin Funk.

For further research

www.imaginary.org/film/glacial-mystery

8c Mappae Mundi

It is impossible to map the earth's surface to a plane world map without distortions. This has already been proved by Carl Friedrich Gauß. For cartographers it is therefore clear that there can not be a perfect map. One has always to decide whether a map needs to be conform to the earth's surface with regard to angle, surface area or distance for a specific purpose. In Mappae Mundi, you can explore and compare different kinds of mapping.

Activities

Start the program by tapping the appropriate item on the screen. Select the different kinds of mappings and view them carefully. Pay attention to the surface areas and outlines of the continents. What are the differences?

In the lower left menu you can select the type of interaction. You can rotate the map, draw in so-called distortion ellipses or draw geodesics and loxodromes. In the following we explain details about the distortion ellipses: If you tap the map with your finger, a so-called distortion ellipse is displayed. A red color indicates a strong distortion at this point. A green color indicates that there are hardly any deviations from the original circle. If the map is conform to the earth's surface with regard to angle (conformal or orthomorphic map), then all the distortion ellipses are circles. If the map is conform with regard to surface area (equal-area map), then all the distortion ellipses have the same surface area. If the map is conform with regard to distance (equidistant map), then all the distortion ellipses have equally long half-axes in the direction of the conformity.

Usually, maps are either conform to distances in the direction of longitude or latitude.

The mathematics behind

Different geometrical laws apply for plane and curved surfaces. For example, the angular sum of a triangle is always 180° in the plane. However, a triangle on a sphere has a larger angular sum. Therefore, if you try to map a triangle from a spherical surface to a plane, you either have to distort the distances of the corners or the angles in the triangle.

In the past, various kinds of mappings have been developed which distort distances and angles in different ways. Consider, for example, the cylindrical projection on the right, which is the basic principle of the Mercator projection. Imagine a cylinder wrapped around the globe of the earth. The cylinder touches the globe at the equator. Starting from the center of the globe, straight lines are drawn through each point of the surface. The corresponding point is depicted where the lines hit the cylinder. Finally, the cylinder is rolled up and a plane map of the earth's surface is obtained. Close to the equator, the map is free of distortion, but the further away from the equator, the greater the distortions of the distances and surface areas.

Instead of the cylinder, other auxiliary bodies can be used, for example cones. This results in other mappings with different distortions.

Acknowledgement

The program has been developed by Dr. Daniel Ramos from the Museu de Matemàtiques de Catalunya.

For further research

www.imaginary.org/program/mappae-mundi

8d Melting of the ice sheets

Due to global warming, sea levels are rising for a variety of reasons. The glaciers of Antarctica and Greenland, known as ice covers, also play a major role. Is it possible to predict changes in the ice covers or even the calving of a glacier?

Activities

This module shows you numerical simulations of ice cover dynamics. It is divided into 12 chapters, similar to an interactive book. Each chapter corresponds to a page and you can click through them one by one. On the last page, a short quiz challenges your knowledge.

The mathematics behind

Physical processes such as gravity, pressure and flow velocity work in the glaciers and are included into the mathematical model with their equations. The model and concrete measurements, namely the glacier height, are the basis for an algorithm to describe the dynamics of a glacier and the entire ice cover. In order to develop a numerical simulation for estimating future changes, the mathematicians use the method of spatial discretization: instead of mapping the entire ice cover with a simulation, they divide the ice cover into smaller units. This increases accuracy because, for example, higher dynamics at the edge of the ice field can be taken into account. In doing so, mathematicians lay a small-scale network over the ice cover and can predict its spatial dynamics in various scenarios over several centuries.

Acknowledgement

The authors of the module are Maëlle Nodet (University Grenoble Alpes) and Jocelyne Erhel (Inria) with the participation of Victoria Denys and supported by Interstices.

For further research

imaginary.github.io/melting-ice-caps/#1

The Power Grid Dynamics Simulation

Power grids are sophisticated and balanced systems to conduct and distribute energy. They are designed in a way, that local disturbances can be quickly compensated and that power grids can operate at a consistent frequency. Mathematicians contribute to the stability of power grids by developing simulations with which critical points within the system can be identified.

Activities

This simulation shows the electricity transmission grid of Scandinavia. The hubs represent the electricity producers or consumers and the lines in between represent the transmission lines. On the right side of the map you can find different sliders and buttons to influence the simulation of the power grid. The blue buttons allow you to stop, start and reset the simulation. If you click on the button "Random State", the program will select a phase shift and frequency of every hub at random. You can also click on a hub and adjust the frequency or the damping to your preferences. The frequency meter on the right below the map indicates how quickly the power grid compensates disturbances. Are you able to interrupt the power supply for an extended time?

The mathematics behind

On the basis of such simulations scientists try to identify which hubs in a power grid are especially vulnerable and which type of disturbances are destabilizing for the system. The mathematical equation behind the simulation considers both the capacity of the transmission lines as well as the power being fed into the grid and consumed. Additionally, it takes into account the condition of every hub which depends on the phase shift and the frequency. At a frequency of 50 Hertz, which is the common frequency of Europe's power grids, the grid is stable. If the frequency is too low, there is a shortage of electricity – if the frequency is too high, there is a surplus of electricity in the grid. Through the simulation of modifications at the hubs, mathematicians can not only detect weaknesses of the system, but also gain valuable knowledge about the future design of grid expansions.

Acknowledgement

This module was developed by Frank Hellmann and Paul Schultz within the Project Condynet, supported by BmBF.

For further research

www.condynet.de/animation.html

9 Double pendulum

You think to know how a pendulum behaves? Think again, our double pendulum blows minds. It makes the funniest movements whose sequence isn't predictable. The analysis of such phenomenon in mathematics went down in history under the term "chaos theory".

Activities

Get the double pendulum in motion using the knob and see what happens. Experiment with different starting points. Try starting twice under the same conditions. How does the pendulum behave?

The mathematics behind

In school, one predominantly learns mechanical or electric systems that demonstrate a simple oscillation behavior, for example a simple pendulum. Its movement is easily predictable: It swings back and forth and stops at a certain point in time. If one knows the air resistance, the frictional forces and further variables, one can mathematically determine the exact motion sequence with the help of a damped sinusoidal oscillation, as well as the magnitude of the deflection and the duration of the pendulum movement.

This does not apply to our double pendulum. Though it is possible to describe the motion mathematically through a system of differential equations, this system of equations can only be solved approximately for certain starting parameters. It is not possible to predict the motion sequence for random starting points in the long-term. Minimal changes can cause enormous differences. This is due to the fact that both pendulums can influence each other and small variations can be reinforced exponentially.

Systems with such behavior are described as "chaotic" systems. Contrary to its common meaning, in mathematics "chaotic" does not mean an absence of order. Chaotic systems are mathematically describable und principally behave in a deterministic way, meaning they are not determined by chance. However, practically it is unbelievably difficult to achieve the same result twice or to predict long-term results since these systems are very vulnerable to small modifications.

In nature, chaotic systems are a common occurrence, such as in meteorology (weather forecasts), astronomy (three-body problem) and in medicine (cardiopulsation rhythm). You probably know the metaphor of a butterfly triggering a tornado in Australia by flapping its wings at Lake Constance, the so-called butterfly effect: Smallest modifications of the starting conditions lead to enormous changes in the result. The mathematical research of such systems is a complex and active research area.

Acknowledgement

This exhibit was planned, designed and built by the firm A2 Metallbau Armbruster in Oberwolfach and sponsored to MiMa.

For further research

www.mima.museum/mathematik-doppelpendel

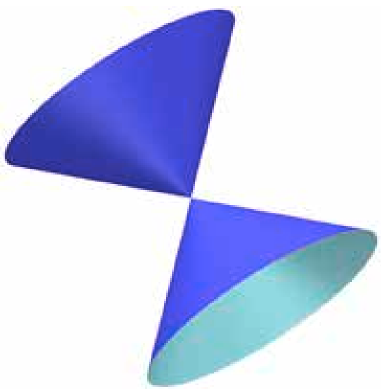
10 SURFER

With SURFER you can experience the relation between formulas and forms in an interactive way. You can enter simple equations that produce beautiful images, which are surfaces in space. Mathematically speaking, SURFER is a program for visualizing real algebraic geometry. The surfaces represent the real solutions of polynomial equations with three variables.

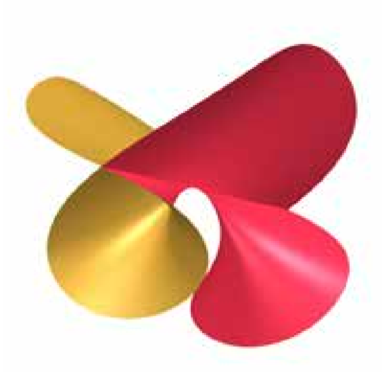
Activities

Enter a polynomial with three variables x, y, and z. The variables may be added and multiplied. Parentheses may also be set according to the rules of mathematics. You can express exponents using the ^ character (for example, write x^2 for x^2).

If your input does not represent a valid polynomial, a red exclamation mark appears next to it. For example, "non-algebraic" expressions of the form 2^x, logarithmic or trigonometric functions are not allowed.



$x^2+y^2-z^2=0$



$x^3+x^2z^2-y^2=0$

SURFER calculates the surface associated with your polynomial and displays it on the screen. By running your finger over it, you can rotate the surface and view it from all sides. Since such surfaces often extend to infinity, SURFER only displays a specific, spherical section of the area. You can use the zoom bar on the right edge of the display to enlarge and reduce this section. With the help of the "Colors" menu item, you can assign different colors to the outside and inside of the surface.

When entering the equation, you can use up to four parameters a, b, c and d (e.g. $ax^2+by^2-3z^2$). A slide bar then appears for each parameter, with which you can change the value of the parameter.

Under the menu item "Start" you will find a large selection of surfaces that you can view or use as a starting point for your own surfaces. For many surfaces additional information is available under the menu item "Info".

What is the formula? This spoon was created in 2008 by Valentina Galata with the help of SURFER.

Experiment with different polynomials. Investigate what changes in the polynomial cause specific changes in the surface. Or consider beforehand what kind of image you want to create and try to enter the appropriate polynomial for it.

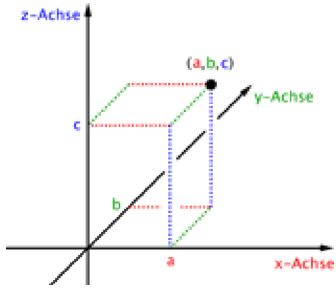
The mathematics behind

Algebraic geometry deals with the interplay between algebra, i.e. the formulas, and geometry, the associated forms.

In our context, a formula is nothing more than a polynomial equation with the three variables x, y, and z, for example $x^2+y^2-z^2 = 0$. Polynomial equation means that on the left-hand side of the equation there is a polynomial. A polynomial is a sum of multiples of powers of variables, where the exponents are natural numbers.

The solutions of the equation $x^2+y^2-z^2 = 0$ are all triples (x,y,z) of numbers for which the left-hand side of the equation equals 0. For example, the triple (3, 4, 5) is a solution to the equation because $3^2+4^2-5^2 = 0$. We say (3, 4, 5) is a "zero" of the polynomial $x^2+y^2-z^2$.

Such triples of numbers can be understood as coordinates of points in three-dimensional space. By choosing three perpendicular coordinate axes, an origin 0 and three directions x, y and z are defined. Each point in three-dimensional space is now uniquely identified by naming three numbers for x, y and z.



With this consideration, the solutions of equations with three variables can be represented as points in space. If one accepts all triples of real numbers as possible solutions, then all solutions or points together result in a surface. An algebraic surface thus visualizes the real solution set of a polynomial equation with three variables.

Algraic geometry can do much more than create beautiful visual shapes. The idea of considering algebraic problems geometrically and vice versa helps to answer important questions in many areas of mathematics. One of the most famous problems in mathematics, the proof of Fermat's last theorem from the 17th century, was solved by Andrew Wiles in 1993/1994 using methods of abstract algebraic geometry. Alexander Grothendieck, among others, had previously made decisive progress in this area. Other applications of algebraic geometry can be found in computer graphics and theoretical physics.

For further research

www.mima.museum/mathematik-surfer

Acknowledgement

SURFER was a project of the MFO and the Technical University of Kaiserslautern. It is based on the Program Surf by Dr. Stephan Endrass and others (surf.sourceforge.net). SURFER was developed under the direction of Prof. Dr. Dr. h.c. Gert Martin Greuel by Dr. Andreas Daniel Matt, Dr. Henning Meyer, Dr. Christian Stussak, Dr. Oliver Labs, Prof. Dr. Herwig Hauser and Felix Riemann.

11 Picture gallery

In the gallery, you can enjoy a selection of images created by various mathematicians and visualization artists.

Dr. Aurélien Alvarez is a researcher at the Université de Lyon. His main interest is in geometry and his main research topics are ergodic and geometric group theory. Together with Étienne Ghys and Jos Leys he worked on the realization of "Dimensions", a film in nine chapters, from stereographic projection and the polyhedron in 4-dimensional space to complex numbers and Hopf fibration.

Luc Benard, a Canadian from Montreal, has worked as a film technician, cinematographer, audio engineer and is currently a video editor. With the increasing development of computers, he began to use fractals as a starting point for his visual work. In recent years he has been working with 3D images and trying to combine science and art in his visual compositions.

Prof. Dr. Étienne Ghys is Director Emeritus of the CNRS (Centre Nationale de la Recherche Scientifique) and works at the Department of Mathematics at the Ecole Normale Supérieure in Lyon. He gave the opening lecture on "Knots and Dynamics" at the International Congress of Mathematicians in Madrid in 2006. Pictures for this lecture were created together with Jos Leys. Leys, Alvarez, and Ghys also collaborated on a series of DVDs on mathematical visualization.

Prof. Dr. Herwig Hauser is Professor of Algebraic Geometry and Singularity Theory at the Department of Mathematics at the University of Vienna. He has been working on visualizations, films and books for a wide audience for many years. Together with his group he organized exhibitions and lectures on algebraic visualization, e.g. his film "Zero Set" was shown at the 2006 International Congress of Mathematicians in Madrid.

Dr. Oliver Labs is a mathematician at Mainz University. For many years he has been working on the construction and visualization of singular algebraic surfaces. Together with co-authors he developed several computer programs on this topic, e.g. Surfex and Surfer. Oliver Labs also runs its own company called MO-Labs, which offers mathematical art, sculptures, jewelry, 3D models and other mathematical objects.

Jos Leys is a graduate engineer. He has always had a great interest in mathematics. Above all, his passion is the creation of mathematical images. Among other things, he coordinates the website "Mathematical Imagery" (www.josleys.com) and has won various awards with it.

Prof. Dr. Richard Palais, Professor Emeritus at Brandeis University began working in the field of mathematical visualization at an early age. He leads an international team, the 3DXM consortium, and is the chief architect and programmer of the software 3D-XplorMath. He is currently at the Department of Mathematics at the University of Irvine in California.

Prof. Dr. Ulrich Pinkall studied mathematics at the University of Freiburg, where he received his doctorate in 1982. Since 1986 he has been a professor for differential geometry and visualization at the Technical University of Berlin. From 1992 to 2003 he was speaker of the Collaborative Research Center "Differential Geometry and Quantum Physics". Since 2004, together with John Sullivan, he has headed the "Mathematical Visualization" working group at the TU Berlin, which is also part of the DFG research center Matheon.

Uli Gaenshirt is a freelance sculptor in Nuremberg. He is interested in mathematical structures in art, since 2001 especially in the mathematics of aperiodic systems and quasi-crystalline structures. He held his first exhibitions between 2008 and 2011 in cooperation with the KOMM education department in Nuremberg. The ornaments shown here provide artistic access to the special geometry of quasicrystals.

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Oberwolfach

Schwarzwaldstr. 9-11
77709 Oberwolfach

+49 (0) 7834 979 0

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